

OUTCOME 4

Circles

1 Representing a Circle

The equation of a circle with centre (a, b) and radius r units is:

$$(x - a)^2 + (y - b)^2 = r^2$$

This is given in the exam.

For example, the equation of a circle with centre $(2, -1)$ and radius 4 is:

$$(x - 2)^2 + (y + 1)^2 = 4^2$$

$$(x - 2)^2 + (y + 1)^2 = 16$$

Note that the equation of a circle with centre $(0, 0)$ is of the form $x^2 + y^2 = r^2$.

EXAMPLE

Find the equation of the circle with centre $(1, -3)$ and radius $\sqrt{3}$ units.

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x - 1)^2 + (y - (-3))^2 = (\sqrt{3})^2$$

$$(x - 1)^2 + (y + 3)^2 = 3$$

2 Testing a Point

Given a circle with centre (a, b) and radius r units, we can determine whether a point (p, q) lies within, outwith or on the circumference using the following rules:

$$(p - a)^2 + (q - b)^2 < r^2 \Leftrightarrow \text{the point lies within the circle}$$

$$(p - a)^2 + (q - b)^2 = r^2 \Leftrightarrow \text{the point lies on the circumference of the circle}$$

$$(p - a)^2 + (q - b)^2 > r^2 \Leftrightarrow \text{the point lies outwith the circle}$$

EXAMPLE

A circle has the equation $(x - 2)^2 + (y + 5)^2 = 29$.

Determine whether the points $(2, 1)$, $(7, -3)$ and $(3, -4)$ lie within, outwith or on the circumference of the circle.

Point $(2, 1)$:

$$\begin{aligned} & (x - 2)^2 + (y + 3)^2 \\ &= (2 - 2)^2 + (1 + 5)^2 \\ &= 0^2 + 6^2 \\ &= 36 > 29 \end{aligned}$$

So outwith the circle.

Point $(7, -3)$:

$$\begin{aligned} & (x - 2)^2 + (y + 3)^2 \\ &= (7 - 2)^2 + (-3 + 5)^2 \\ &= 5^2 + 2^2 \\ &= 29 \end{aligned}$$

So on the circumference.

Point $(3, -4)$:

$$\begin{aligned} & (x - 2)^2 + (y + 3)^2 \\ &= (3 - 2)^2 + (-4 + 5)^2 \\ &= 1^2 + 1^2 \\ &= 2 < 29 \end{aligned}$$

So within the circle.

3 The General Equation of a Circle

The equation of a circle can be written in the form:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where the centre is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$ units.

This is given in the exam.

Note that the above equation only represents a circle if $g^2 + f^2 - c > 0$, since:

- if $g^2 + f^2 - c < 0$ then we cannot obtain a real value for the radius
- if $g^2 + f^2 - c = 0$ then the radius is zero – the equation represents a point.

EXAMPLE

1. Find the radius and centre of the circle with equation

$$x^2 + y^2 + 4x - 8y + 7 = 0.$$

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$\begin{array}{lll} 2g = 4 \text{ so } g = 2 & \text{Centre is } (-g, -f) & \text{Radius is } \sqrt{g^2 + f^2 - c} \\ 2f = -8 \text{ so } f = -4 & = (-2, 4) & = \sqrt{2^2 + (-4)^2 - 7} \\ c = 7 & & = \sqrt{4 + 16 - 7} \\ & & = \sqrt{13} \text{ units} \end{array}$$

2. Find the radius and centre of the circle with equation

$$2x^2 + 2y^2 - 6x + 10y - 2 = 0.$$

The equation must be in the form $x^2 + y^2 + 2gx + 2fy + c = 0$, so divide each term by 2:

$$x^2 + y^2 - 3x + 5y - 1 = 0$$

Now compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$\begin{aligned} 2g = -3 \text{ so } g = -\frac{3}{2} & \quad \text{Centre is } (-g, -f) & \quad \text{Radius is } \sqrt{g^2 + f^2 - c} \\ 2f = 5 \text{ so } f = \frac{5}{2} & \quad = \left(\frac{3}{2}, -\frac{5}{2}\right) & \quad = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 1} \\ c = -1 & & \quad = \sqrt{\frac{9}{4} + \frac{25}{4} - \frac{4}{4}} \\ & & \quad = \sqrt{\frac{38}{4}} \\ & & \quad = \frac{\sqrt{38}}{2} \text{ units} \end{aligned}$$

3. Explain why $x^2 + y^2 + 4x - 8y + 29 = 0$ is not the equation of a circle.

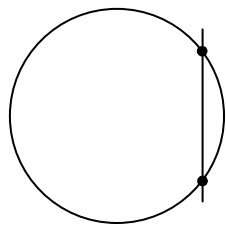
Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$\begin{aligned} 2g = 4 \text{ so } g = 2 & & g^2 + f^2 - c = 2^2 + (-4)^2 - 29 \\ 2f = -8 \text{ so } f = -4 & & = -9 < 0 \\ c = 29 & & \end{aligned}$$

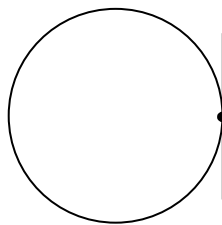
The equation does not represent a circle since $g^2 + f^2 - c > 0$ is not satisfied.

4 Intersection of a Line and a Circle

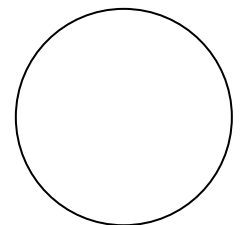
A straight line and circle can have two, one or no points of intersection:



two intersections



one intersection



no intersections

If a line and a circle only touch at one point, then the line is a tangent to the circle at that point. To find out how many times a line and circle meet, we can use substitution.

EXAMPLES

1. Find the points where the line with equation $y = 3x$ intersects the circle with equation $x^2 + y^2 = 20$.

$$x^2 + y^2 = 20$$

$$x^2 + (3x)^2 = 20$$

$$x^2 + 9x^2 = 20$$

$$10x^2 = 20$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$x = \sqrt{2} \Rightarrow y = 3(\sqrt{2}) = 3\sqrt{2} \quad x = -\sqrt{2} \Rightarrow y = 3(-\sqrt{2}) = -3\sqrt{2}$$

So the circle and the line meet at $(\sqrt{2}, 3\sqrt{2})$ and $(-\sqrt{2}, -3\sqrt{2})$.

Remember

$$(ab)^m = a^m b^m$$

2. Find the points where the line with equation $y = 2x + 6$ and circle with equation $x^2 + y^2 + 2x + 2y - 8 = 0$ intersect.

Substitute $y = 2x + 6$ into the equation of the circle:

$$x^2 + (2x + 6)^2 + 2x + 2(2x + 6) - 8 = 0$$

$$x^2 + (2x + 6)(2x + 6) + 2x + 4x + 12 - 8 = 0$$

$$x^2 + 4x^2 + 24x + 36 + 2x + 4x + 12 - 8 = 0$$

$$5x^2 + 30x + 40 = 0$$

$$5(x^2 + 6x + 8) = 0$$

$$(x + 2)(x + 4) = 0$$

$$x + 2 = 0$$

$$x + 4 = 0$$

$$x = -2 \Rightarrow y = 2(-2) + 6 = 2$$

$$x = -4 \Rightarrow y = 2(-4) + 6 = -2$$

So the line and circle meet at $(-2, 2)$ and $(-4, -2)$.

5 Tangents to Circles

As we have seen a line is a tangent if it intersects the circle at only one point.

To show that a line is a tangent to a circle, the equation of the line can be substituted into the equation of the circle, and solved – there should only be one solution.

EXAMPLE

Show that the line with equation $x + y = 4$ is a tangent to the circle with equation $x^2 + y^2 + 6x + 2y - 22 = 0$.

Substitute y for equation of straight line:

$$x^2 + y^2 + 6x + 2y - 22 = 0$$

$$x^2 + (4 - x)^2 + 6x + 2(4 - x) - 22 = 0$$

$$x^2 + (4 - x)(4 - x) + 6x + 2(4 - x) - 22 = 0$$

$$x^2 + 16 - 8x + x^2 + 6x + 8 - 2x - 22 = 0$$

$$2x^2 - 4x + 2 = 0$$

$$2(x^2 - 2x + 1) = 0$$

$$x^2 - 2x + 1 = 0$$

Then (i) factorise or (ii) use the discriminant

$$x^2 - 2x + 1 = 0$$

$$(x - 1)(x - 1) = 0$$

$$x - 1 = 0 \quad x - 1 = 0$$

$$x = 1 \quad x = 1$$

Since the solutions are equal, the line is a tangent to the circle.

$$x^2 - 2x + 1 = 0$$

$$a = 1 \quad b^2 - 4ac$$

$$b = -2 \quad = (-2)^2 - 4(1 \times 1)$$

$$c = 1 \quad = 4 - 4$$

$$= 0$$

Since $b^2 - 4ac = 0$, the line is a tangent to the circle.

Note

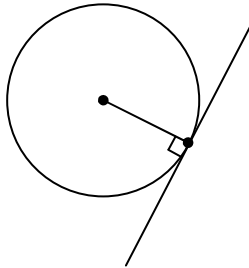
If the point of contact is required then method (i) is more efficient.

To find the point, substitute the value found for x into the equation of the line (or circle) to calculate the corresponding y -coordinate.

6 Equations of Tangents to Circles

If the point of contact between a circle and a tangent is known, then the equation of the tangent can be calculated.

If a line is a tangent to a circle, then the radius will meet the tangent at right angles. The gradient of this radius can be calculated, since the centre and point of contact are known.



Using $m_{\text{radius}} \times m_{\text{tangent}} = -1$, the gradient of the tangent can be found.

The equation can then be found using $y - b = m(x - a)$, since the point is known, and the gradient has just been calculated.

EXAMPLE

Show that $A(1, 3)$ lies on the circle $x^2 + y^2 + 6x + 2y - 22 = 0$ and find the equation of the tangent at A .

Substitute point into equation of circle:

$$\begin{aligned} x^2 + y^2 + 6x + 2y - 22 \\ = 1^2 + 3^2 + 6(1) + 2(3) - 22 \\ = 1 + 9 + 6 + 6 - 22 \\ = 0 \end{aligned}$$

Since this satisfies the equation of the circle, the point must lie on the circle.

Find gradient between the centre $(-3, -1)$ and point $(1, 3)$:

$$\begin{aligned} m_{\text{radius}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3 + 1}{1 + 3} \\ &= 1 \quad \text{so } m_{\text{tangent}} = -1 \quad \text{since } m_{\text{radius}} \times m_{\text{tangent}} = -1 \end{aligned}$$

Find equation of tangent using point $(1, 3)$ and gradient $m = -1$:

$$\begin{aligned} y - b &= m(x - a) \\ y - 3 &= -(x - 1) \\ y - 3 &= -x + 1 \\ y &= -x + 4 \\ x + y - 4 &= 0 \end{aligned}$$

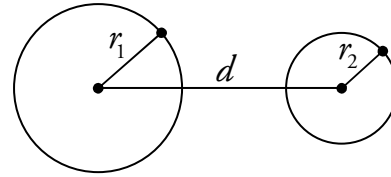
Therefore the equation of the tangent to the circle at A is $x + y - 4 = 0$.

7 Intersection of Circles

Consider two circles with radii r_1 and r_2 with $r_1 > r_2$.

Let d be the distance between the centres of the two circles.

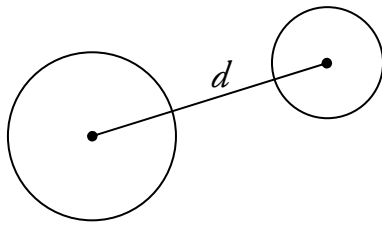
There are three cases to consider.



The circles
do not touch

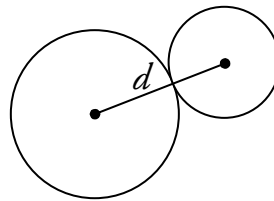
They meet at
one point only

They meet at
two distinct points

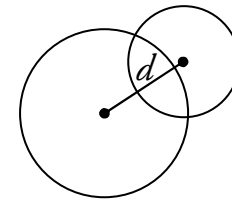


$$d > r_1 + r_2$$

Externally

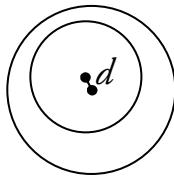


$$d = r_1 + r_2$$

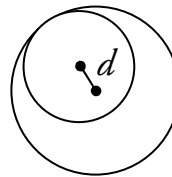


$$r_1 - r_2 < d < r_1 + r_2$$

Internally



$$d < r_1 - r_2$$



$$d = r_1 - r_2$$

EXAMPLES

1. Circle P has centre $(-4, -1)$ and radius 2 units, circle Q has equation $x^2 + y^2 - 2x + 6y + 1 = 0$. Show that the circles P and Q do not touch.



To find the centre and radius of Q:

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$\begin{aligned} 2g = -2 \text{ so } g = -1 & \quad \text{Centre is } (-g, -f) & \quad \text{Radius } r_Q = \sqrt{g^2 + f^2 - c} \\ 2f = 6 \text{ so } f = 3 & \quad = (1, -3) & \quad = \sqrt{1 + 9 - 1} \\ c = 1 & & \quad = \sqrt{9} \\ & & \quad = 3 \text{ units} \end{aligned}$$

We know P has centre $(-4, -1)$ and radius $r_P = 2$ units.

$$\begin{aligned} \text{So the distance between the centres } d &= \sqrt{(1+4)^2 + (-3+1)^2} \\ &= \sqrt{5^2 + (-2)^2} \\ &= \sqrt{29} \approx 5.39 \text{ units} \end{aligned}$$

Since $r_P + r_Q = 3 + 2 = 5 < d$, the circles P and Q do not touch.

2. Circle R has equation $x^2 + y^2 - 2x - 4y - 4 = 0$, and circle S has equation $(x - 4)^2 + (y - 6)^2 = 4$. Show that the circles R and S touch externally.

To find the centre and radius of R:

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$\begin{array}{llll}
 2g = -2 \text{ so } g = -1 & \text{Centre is } (-g, -f) & \text{Radius } r_R = \sqrt{g^2 + f^2 - c} \\
 2f = -4 \text{ so } f = -2 & = (1, 2) & = \sqrt{(-1)^2 + (-2)^2 - 4} \\
 c = -4 & & = \sqrt{9} \\
 & & = 3 \text{ units}
 \end{array}$$

To find the centre and radius of S:

Compare with $(x - a)^2 + (y - b)^2 = r^2$:

$$\begin{array}{llll}
 a = 4 & \text{Centre is } (a, b) & \text{Radius } r_S = 2 \text{ units} \\
 b = 6 & = (4, 6) & \\
 r^2 = 4 \text{ so } r = 2 & &
 \end{array}$$

$$\begin{aligned}
 \text{So the distance between the centres } d &= \sqrt{(1 - 4)^2 + (2 - 6)^2} \\
 &= \sqrt{(-3)^2 + (-4)^2} \\
 &= \sqrt{25} \\
 &= 5 \text{ units}
 \end{aligned}$$

Since $r_R + r_S = 3 + 2 = 5 = d$, the circles R and S touch externally.