OUTCOME 1

Polynomials and Quadratics

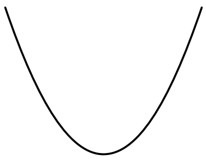
1 Quadratics

A quadratic has the form $ax^2 + bx + c$ where a, b, and c are any numbers, provided $a \neq 0$.

You should already be familiar with the following.

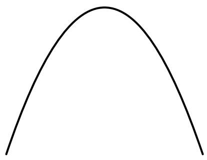
The graph of a quadratic is called a **parabola**. There are two possible shapes:

concave up (if $a > 0$)



This has a minimum turning point

concave down (if $a < 0$)



This has a maximum turning point

To find the roots of the quadratic $ax^2 + bx + c = 0$, we can use:

- factorisation
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (this is **not** given in the exam)

EXAMPLES

1. Find the roots of $x^2 - 2x - 3 = 0$.

$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x+1 = 0 \quad \text{or} \quad x-3 = 0$$
$$x = -1 \qquad\qquad x = 3$$

2. Solve $x^2 + 8x + 16 = 0$.

$$x^2 + 8x + 16 = 0$$
$$(x+4)(x+4) = 0$$
$$x+4 = 0 \quad \text{or} \quad x+4 = 0$$
$$x = -4 \qquad\qquad x = -4$$

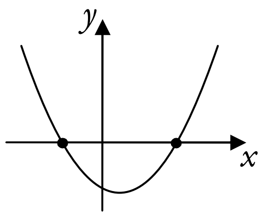
3. Find the roots of $x^2 + 4x - 1 = 0$.

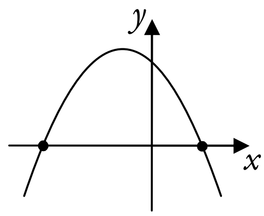
We cannot factorise $x^2 + 4x - 1$, so we use the quadratic formula:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\ &= \frac{-4 \pm \sqrt{16 + 4}}{2} \\ &= \frac{-4 \pm \sqrt{20}}{2} \\ &= -\tfrac{4}{2} \pm \tfrac{\sqrt{4}\sqrt{5}}{2} \\ &= -2 \pm \sqrt{5} \end{aligned}$$

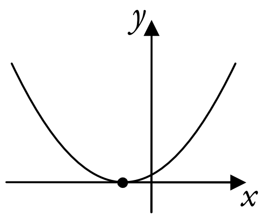
Note

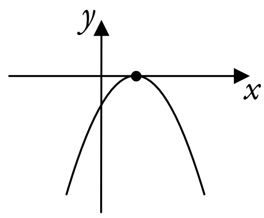
- If there are two distinct solutions, the curve intersects the x-axis twice.



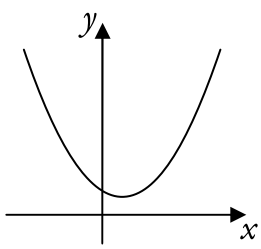


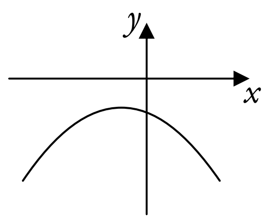
- If there is one repeated solution, the turning point lies on the x-axis.





- If $b^2 - 4ac < 0$ when using the quadratic formula, there are no points where the curve intersects the x-axis.



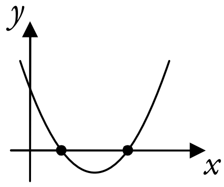


2 The Discriminant

Given $ax^2 + bx + c$, we call $b^2 - 4ac$ the discriminant.

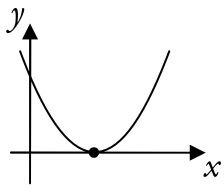
It is the part of the quadratic formula which determines how many real roots $ax^2 + bx + c = 0$ has.

- If $b^2 - 4ac > 0$, the roots are real and unequal (distinct).



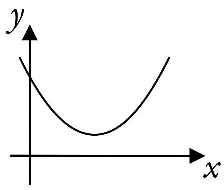
two roots

- If $b^2 - 4ac = 0$, the roots are real and equal (i.e. a repeated root).



one root

- If $b^2 - 4ac < 0$, the roots are not real; they do not exist.



no real roots

EXAMPLE

1. Find the nature of the roots of $9x^2 + 24x + 16 = 0$.

$a = 9 \qquad b^2 - 4ac = 24^2 - 4 \times 9 \times 16$

$b = 24 \qquad\qquad = 576 - 576$

$c = 16 \qquad\qquad = 0$

Since $b^2 - 4ac = 0$, the roots are real and equal.

Using the Discriminant

The discriminant has many uses, usually involving an unknown term in a quadratic.

EXAMPLES

2. Find the values of q such that $6x^2 + 12x + q = 0$ has real roots.

Since $6x^2 + 12x + q = 0$ has real roots, $b^2 - 4ac \geq 0$

$a = 6 \qquad b^2 - 4ac \geq 0$

$b = 12 \qquad 12^2 - 4 \times 6 \times q \geq 0$

$c = q \qquad 144 - 24q \geq 0$

$$144 \geq 24q$$

$$24q \leq 144$$

$$q \leq 6$$

3. Show that $(2k+4)x^2+(3k+2)x+(k-2)=0$ always has real roots.

$a=2k+4$
$b=3k+2$
$c=k-2$

$$\begin{aligned}&b^2-4ac\\&=(3k+2)^2-4(2k+4)(k-2)\\&=9k^2+12k+4-(2k+4)(4k-8)\\&=9k^2+12k+4-8k^2+32\\&=k^2+12k+36\\&=(k+6)^2\end{aligned}$$

Since $b^2-4ac=(k+6)^2\geq 0$, the roots will always be real.

3 Completing the Square

Completing the square involves expressing $y=ax^2+bx+c$ in the form $y=a(x+p)^2+q$.

In this form, we can determine the turning point of any parabola, including those with no real roots.

The axis of symmetry is $x=-p$ and the turning point is $(-p,q)$.

The following process can be used to complete the square:

1. Make sure the equation is in the form $y=ax^2+bx+c$.
2. Separate the constant term (c) by bracketing everything else on the RHS
3. Take out any **constant** common factor from the brackets.
4. Add on half of the x coefficient squared and subtract half of the x coefficient squared (note that adding on a number and taking it away keeps the equation balanced).
5. Factorise the first three terms in the bracket.
6. State your answer in the form $y=a(x+p)^2+q$.

EXAMPLES

1. Write $y=x^2+6x-5$ in the form $y=(x+p)^2+q$.

$$\begin{aligned}y&=(x^2+6x)-5 &&\text{(Steps 1-3)}\\&=(x^2+6x+3^2-3^2)-5 &&\text{(Step 4)}\\&=(x+3)^2-9-5 &&\text{(Step 5)}\\&=(x+3)^2-14 &&\text{(Step 6)}\end{aligned}$$

Note
You can always check your answer by expanding the brackets

2. Write $y = x^2 + 3x - 4$ in the form $y = (x + p)^2 + q$.

$$\begin{aligned} y &= \left(x^2 + 3x\right) - 4 \\ &= \left(x^2 + 3x + \left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right) - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{25}{4} \end{aligned}$$

3. Write $y = x^2 + 8x - 3$ in the form $y = (x + a)^2 + b$ and then state:
(i) the axis of symmetry, and
(ii) the minimum turning point of the parabola with this equation.

$$\begin{aligned} y &= \left(x^2 + 8x\right) - 3 \\ &= \left(x^2 + 8x + 4^2 - 4^2\right) - 3 \\ &= (x + 4)^2 - 16 - 3 \\ &= (x + 4)^2 - 19 \end{aligned}$$

(i) The axis of symmetry is $x = -4$.
(ii) The minimum turning point is $(-4, -19)$.

4. A parabola has equation $y = 4x^2 - 12x + 7$. Express the equation in the form $y = (x + a)^2 + b$, and state the turning point of the parabola.

$$\begin{aligned} y &= \left(4x^2 - 12x\right) + 7 \\ &= 4\left(x^2 - 3x\right) + 7 \\ &= 4\left(x^2 - 3x + \left(-\tfrac{3}{2}\right)^2 - \left(-\tfrac{3}{2}\right)^2\right) + 7 \\ &= 4\left(\left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4}\right) + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 9 + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 2 \end{aligned}$$

The turning point is $\left(\tfrac{3}{2}, -2\right)$.

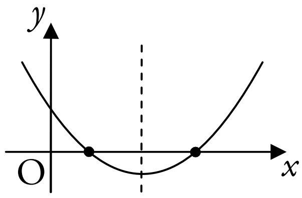
4 Sketching Parabolas

The method used to sketch the curve with equation $y = ax^2 + bx + c$ depends on how many times the curve intersects the x-axis.

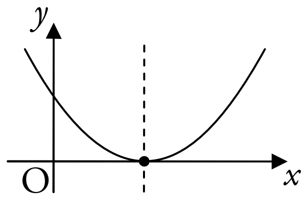
We have met curve sketching before. However, when sketching parabolas, we **do not** need to use calculus. We know there is only one turning point, and we have methods for finding it.

Parabolas with one or two roots

- Find the x-axis intercepts by factorising or using the quadratic formula.
- Find the y-axis intercept (i.e. where $x = 0$).
- The turning point is on the axis of symmetry:



The axis of symmetry is halfway between two distinct roots



A repeated root lies on the axis of symmetry

Parabolas with no real roots

- There are no x-axis intercepts.
- Find the y-axis intercept (i.e. where $x = 0$).
- Find the turning point by completing the square.

EXAMPLES

1. Sketch the graph of $y = x^2 - 8x + 7$.

Since $b^2 - 4ac = (-8)^2 - 4 \times 1 \times 7 > 0$, the parabola crosses the x-axis twice.

The y-axis intercept $(x = 0)$:

$$y = (0)^2 - 8(0) + 7$$
$$= 7$$
$$(0, 7)$$

The x-axis intercepts $(y = 0)$:

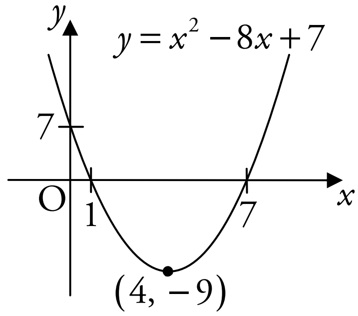
$$x^2 - 8x + 7 = 0$$
$$(x-1)(x-7) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 7 = 0$$
$$x = 1 \qquad\qquad x = 7$$
$$(1, 0) \qquad\qquad (7, 0)$$

The axis of symmetry lies halfway between $x = 1$ and $x = 7$, i.e. $x = 4$, so the x-coordinate of the turning point is 4.

We can now find the y-coordinate:

$$y = (4)^2 - 8(4) + 7$$
$$= 16 - 32 + 7$$
$$= -9$$

So the turning point is $(4,\ -9)$.



2. Sketch the parabola with equation $y = -x^2 - 6x - 9$

Since $b^2 - 4ac = (-6)^2 - 4\times(-1)\times(-9) = 0$, there is a repeated root.

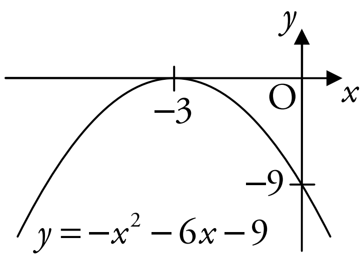
The y-axis intercept $(x = 0)$:

$$y = -(0)^2 - 6(0) - 9$$
$$= -9$$
$$(0,\ -9)$$

The x-axis intercept $(y = 0)$:

$$-x^2 - 6x - 9 = 0$$
$$-(x^2 + 6x + 9) = 0$$
$$(x+3)(x+3) = 0$$
$$x + 3 = 0$$
$$x = -3$$
$$(-3, 0)$$

Since there is a repeated root, $(-3, 0)$ is the turning point.



3. Sketch the curve with equation $y = 2x^2 - 8x + 13$.

Since $b^2 - 4ac = (-8)^2 - 4\times2\times13 < 0$, there are no real roots.

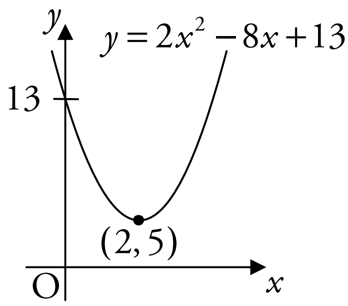
The y-axis intercept $(x = 0)$:

$$y = 2(0)^2 - 8(0) + 13$$
$$= 13$$
$$(0, 13)$$

Complete the square:

$$y = 2x^2 - 8x + 13$$
$$= 2(x^2 - 4x) + 13$$
$$= 2(x-2)^2 - 8 + 13$$
$$= 2(x-2)^2 + 5$$

So the turning point is $(2, 5)$.



5 Determining the Equation of a Parabola

In order to determine the equation of a parabola, we need two pieces of information:

- The x-axis intercept(s) or turning point
- Another point on the parabola

EXAMPLES

1. A parabola passes through the points $(1, 0)$, $(5, 0)$ and $(0, 3)$.
   Find the equation of the parabola.

Since the parabola cuts the x-axis where $x = 1$ and $x = 5$, the equation is of the form:

$$y = k(x-1)(x-5)$$

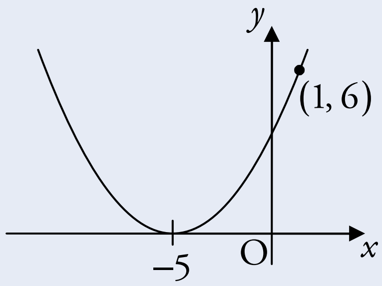
To find k, we use the point $(0, 3)$:

$$y = k(x-1)(x-5)$$
$$3 = k(0-1)(0-5)$$
$$3 = 5k$$
$$k = \tfrac{3}{5}$$

So the equation of the parabola is:

$$y = \tfrac{3}{5}(x-1)(x-5)$$
$$= \tfrac{3}{5}\left(x^2 - 5x - x + 5\right)$$
$$= \tfrac{3}{5}x^2 - \tfrac{18}{5}x + 3$$

2. Find the equation of the parabola shown below.



Since there is a repeated root, the equation is of the form:

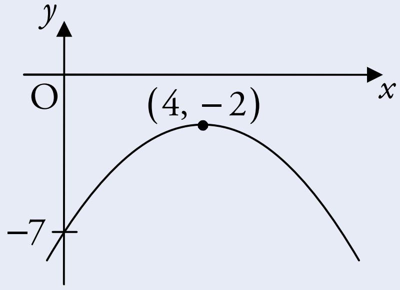
$$y = k(x+5)^2$$

To find k, we use $(1, 6)$:

$$y = k(x+5)^2$$
$$6 = k(1+5)^2$$
$$k = \tfrac{6}{6^2} = \tfrac{1}{6}$$

Hence $y = \tfrac{1}{6}(x+5)^2$.

3. Find the equation of the parabola shown below.



Since the turning point is $(4, -2)$, the equation is of the form:

$$y = a(x-4)^2 - 2$$

To find a, we use $(0, -7)$:

$$y = a(x-4)^2 - 2$$
$$-7 = a(0-4)^2 - 2$$
$$16a = -5$$
$$a = -\tfrac{5}{16}$$

Hence $y = -\frac{5}{16}(x-4)^2 - 2$.

6 Solving Quadratic Inequalities

The most efficient way of solving a quadratic inequality is by making a rough sketch of the parabola. To do this we need to know:

- the shape – concave up or concave down
- the x-axis intercepts.

We can then solve the quadratic inequality by inspection of the sketch.

EXAMPLES

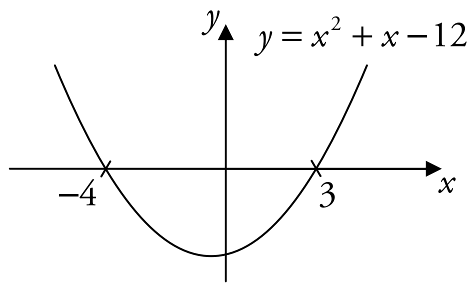
1. Solve $x^2 + x - 12 < 0$.

The parabola with equation $y = x^2 + x - 12$ is concave up.

The x-axis intercepts are given by:

$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -4 \qquad\qquad x = 3$$

Make a sketch:



So $x^2 + x - 12 < 0$ for $-4 < x < 3$.

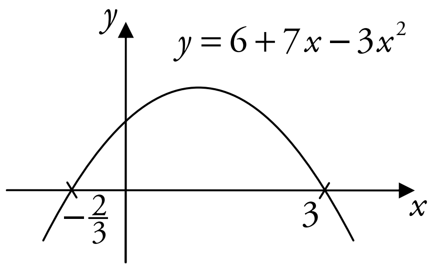
2. Find the values of x for which $6+7x-3x^2 \geq 0$.

The parabola with equation $y=6+7x-3x^2$ is concave down.

The x-axis intercepts are given by:

$$6+7x-3x^2=0$$
$$-(3x^2-7x-6)=0$$
$$(3x+2)(x-3)=0$$
$$3x+2=0 \quad \text{or} \quad x-3=0$$
$$x=-\tfrac{2}{3} \qquad x=3$$

Make a sketch:



So $6+7x-3x^2 \geq 0$ for $-\frac{2}{3} \leq x \leq 3$.

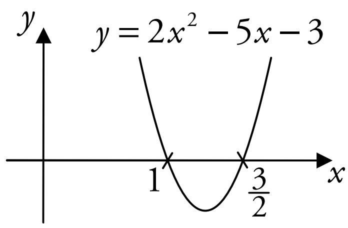
3. Solve $2x^2-5x-3>0$.

The parabola with equation $y=2x^2-5x-3$ is concave up.

The x-axis intercepts are given by:

$$2x^2-5x-3=0$$
$$(x-1)(2x-3)=0$$
$$x-1=0 \quad \text{or} \quad 2x-3=0$$
$$x=1 \qquad x=\tfrac{3}{2}$$

Make a sketch:



So $2x^2-5x-3>0$ for $x<1$ and $x>\frac{3}{2}$.

4. Find the values of q for which $x^2+(q-4)x+\frac{1}{2}q=0$ has non-real roots.

Since $x^2+(q-4)x+\frac{1}{2}q=0$ has non-real roots, $b^2-4ac<0$:

$a=1$

$b=q-4$

$c=\frac{1}{2}q$

$$\begin{aligned} b^2-4ac &= (q-4)^2-4(1)\left(\tfrac{1}{2}q\right) \\ &= (q-4)(q-4)-2q \\ &= q^2-8q+16-2q \\ &= q^2-10q+16 \end{aligned}$$

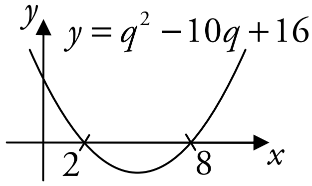
We now need to solve the inequality $q^2-10q+16<0$:

The parabola with equation $y=q^2-10q+16$ is concave up.

The x-axis intercepts are given by:

$$q^2 - 10q + 16 = 0$$
$$(q-2)(q-8) = 0$$
$$q - 2 = 0 \quad \text{or} \quad q - 8 = 0$$
$$q = 2 \qquad\qquad q = 8$$

Make a sketch:

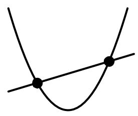


Therefore $b^2 - 4ac < 0$ for $2 < q < 8$, and so $x^2 + (q-4)x + \frac{1}{2}q = 0$ has non-real roots when $2 < q < 8$.

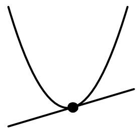
7 Intersections of Lines and Parabolas

To determine how many times a line intersects a parabola, we substitute the equation of the line into the equation of the parabola. We can then use the discriminant or factorisation to find the number of intersections.

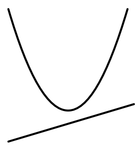
- If $b^2 - 4ac > 0$, the line and curve intersect twice



- If $b^2 - 4ac = 0$, the line and curve intersect once (i.e. the line is a tangent to the curve)



- If $b^2 - 4ac < 0$, the line and the parabola do not intersect



EXAMPLES

1. Show that the line $y = 5x - 2$ is a tangent to the parabola $y = 2x^2 + x$ and find the point of contact.

Substitute $y = 5x - 2$ into:

$$y = 2x^2 + x$$
$$5x - 2 = 2x^2 + x$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)(x-1) = 0$$

Since there is a repeated root, the line is a tangent at $x = 1$.

To find the y-coordinate, substitute $x = 1$ into the equation of the line:

$$y = 5(1) - 2$$
$$y = 3$$

So the point of contact is $(1, 3)$.

2. Find the equation of the tangent to $y = x^2 + 1$ that has gradient 3.

The equation of the tangent is of the form $y = mx + c$, with $m = 3$, i.e. $y = 3x + c$.

Substitute this into:

$$\begin{aligned} y &= x^2 + 1 \\ x^2 + 1 &= 3x + c \\ x^2 - 3x + 1 - c &= 0 \end{aligned}$$

Since the line is a tangent:

$$\begin{aligned} b^2 - 4ac &= 0 \\ (-3)^2 - 4 \times (1 - c) &= 0 \\ 9 - 4 + 4c &= 0 \\ 4c &= -5 \\ c &= -\tfrac{5}{4} \end{aligned}$$

Therefore the equation of the tangent is:

$$\begin{aligned} y &= 3x - \tfrac{5}{4} \\ 3x - y - \tfrac{5}{4} &= 0 \end{aligned}$$

8 Polynomials

Polynomials are expressions with one or more terms added together. Each term has a number (called the **coefficient**) followed by a variable (such as x) raised to a whole number power. For example:

$$3x^5 + x^3 + 2x^2 - 6 \quad \text{or} \quad 2x^{18} + 10$$

The **degree** of the polynomial is the value of its highest power, for example:

$3x^5 + x^3 + 2x^2 - 6$ has degree 5 $\qquad$ $2x^{18} + 10$ has degree 18

Note that quadratics are polynomials of degree two. Also, constants are polynomials of degree zero (e.g. 6 is a polynomial, since $6 = 6x^0$).

9 Synthetic Division

Synthetic division provides a quick way of evaluating functions defined by polynomials.

For example, consider $f(x) = 2x^3 - 9x^2 + 2x + 1$. Using a calculator, we find $f(6) = 121$. We can also evaluate this using synthetic division with detached coefficients.

Step 1

Detach the coefficients, and write them across the top row of the table.

Note that they must be in order of **decreasing** degree. If there is no term of a specific degree, then zero is its coefficient.

	2	–9	2	1

Step 2

Write the number for which you want to evaluate the polynomial (the input number).

6	2	–9	2	1

Step 3

Bring down the first coefficient.

6	2	–9	2	1
	2			

Step 4

Multiply this by the input number, writing the result underneath the next coefficient.

6	2	–9	2	1
		12		
	2			

Step 5

Add the numbers in this column.

6	2	–9	2	1
		12		
	2	3		

Repeat Steps 4 and 5 until the last column has been completed.

The number in the lower-right cell is the value of the polynomial for the input value, often referred to as the **remainder**.

6	2	–9	2	1
		12	18	120
	2	3	20	121

EXAMPLE

1. Given $f(x)=x^3+x^2-22x-40$, evaluate $f(-2)$ using synthetic division.

–2	1	1	–22	–40

using the above process →

–2	1	1	–22	–40
		–2	2	40
	1	–1	–20	0

So $f(-2)=0$.

Note

In this example, the remainder is zero, so $f(-2)=0$.

This means $x^3+x^2-22x-40=0$ when $x=-2$, which means that $x=-2$ is a root of the equation. So $x+2$ is a factor of the cubic.

We can use this to help with factorisation:

$$f(x)=(x+2)(\ q(x)\ ) \qquad \text{where } q(x) \text{ is a quadratic}$$

Is it possible to find the quadratic $q(x)$ using the table?

Trying the numbers from the bottom row as coefficients, we find:

$$\begin{aligned}&(x+2)(x^2-x-20)\\&=x^3-x^2-20x+2x^2-2x-40\\&=x^3-x^2-22x-40\\&=f(x)\end{aligned}$$

–2	1	1	–22	–40
		–2	2	40
	1	–1	–20	0

So using the numbers from the bottom row as coefficients does give the correct quadratic.

EXAMPLES

2. Show that $x-4$ is a factor of $2x^4-9x^3+5x^2-3x-4$.

$x-4$ is a factor $\Leftrightarrow$ $x=4$ is a root.

4	2	–9	5	–3	–4
		8	–4	4	4
	2	–1	1	1	0

Since the remainder is zero, $x=4$ is a root, so $x-4$ is a factor.

3. Given $f(x) = x^3 - 37x + 84$
   (i) show that $x = -7$ is a root of $f(x) = 0$, and
   (ii) hence fully factorise $f(x)$.

-7	1	0	-37	84
		-7	49	-84
	1	-7	12	0

Since the remainder is zero, $x = -7$ is a root.

Hence we have
$$\begin{aligned} f(x) &= x^3 - 37x + 84 \\ &= (x+7)(x^2 - 7x + 12) \\ &= (x+7)(x-3)(x-4) \end{aligned}$$

4. Show that $x = -5$ is a root of $2x^3 + 7x^2 - 9x + 30 = 0$, and hence fully factorise the cubic.

-5	2	7	-9	30
		-10	15	-30
	2	-3	6	0

Since $x = -5$ is a root, $x + 5$ is a factor.

$2x^3 + 7x^2 - 9x + 30 = (x+5)(2x^2 - 3x + 6)$

This does not factorise any further since $b^2 - 4ac < 0$.

Using synthetic division to factorise

In the examples above, we have been given a root or factor to help factorise polynomials. However, we can still use synthetic division if we do not know a factor or root.

The roots of the polynomial $f(x)$ are numbers which divide the constant term exactly. So by trying synthetic division with these numbers, we will (eventually) find a root.

5. Fully factorise $2x^3 + 5x^2 - 28x - 15$.

Numbers which divide -15: $\pm1, \pm3, \pm5, \pm15$.

Try $x = 1$: $2(1)^3 + 5(1)^2 - 28(1) - 15$
$= 2 + 5 - 28 - 15 \neq 0$

Try $x = -1$: $2(-1)^3 + 5(-1)^2 - 28(-1) - 15$
$= -2 + 5 + 28 - 15 \neq 0$

Note
For ±1, it is simpler just to evaluate the polynomial directly, to see if this is a root

Try $x = 3$:

3	2	5	–28	–15
		6	33	15
	2	11	5	0

Since $x = 3$ is a root, $x - 3$ is a factor.

So $2x^3 + 5x^2 - 28x - 15 = (x-3)(2x^2 + 11x + 5)$

$= (x-3)(2x+1)(x+5)$

Using synthetic division to solve equations

We can also use synthetic division to help solve equations.

EXAMPLE

6. Find the roots of $2x^3 - 15x^2 + 16x + 12 = 0$.

Numbers which divide 12: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

Try $x = 1$: $2(1)^3 - 15(1)^2 + 16(1) + 12$

$= 2 - 15 + 16 + 12 \neq 0$

Try $x = -1$: $2(-1)^3 - 15(-1)^2 + 16(-1) + 12$

$= -2 - 15 - 16 + 12 \neq 0$

Try $x = 2$:

2	2	–15	16	12
		4	–22	–12
	2	–11	–6	0

Since $x = 2$ is a root, $x - 2$ is a factor:

$$2x^3 - 15x^2 + 16x + 12 = 0$$
$$(x-2)(2x^2 - 11x - 6) = 0$$
$$(x-2)(2x+1)(x-6) = 0$$

$x - 2 = 0$ or $2x + 1 = 0$ or $x - 6 = 0$

$x = 2$ $\quad x = -\frac{1}{2}$ $\quad x = 6$

In general

For a polynomial $f(x)$:

If $f(x)$ is divided by $x - h$ then the remainder is $f(h)$, and

$$f(h) = 0 \Leftrightarrow x - h \text{ is a factor of } f(x).$$

As we saw, synthetic division helps us to write $f(x)$ in the form $(x-h)q(x) + f(h)$ where $q(x)$ is called the **quotient**.

EXAMPLE

7. Find the quotient and remainder when $f(x) = 4x^3 + x^2 - x - 1$ is divided by $x+1$, and express $f(x)$ as $(x+1)q(x) + f(h)$.

$$\begin{array}{r|rrrr} -1 & 4 & 1 & -1 & -1 \\ & & -4 & 3 & -2 \\ \hline & 4 & -3 & 2 & \boxed{-3} \end{array}$$

The quotient is $4x^2 - 3x + 2$ and the remainder is –3, so

$$f(x) = (x+1)(4x^2 - 3x + 2) - 3$$

10 Finding Unknown Coefficients

Consider a polynomial with some unknown coefficients, such as $x^3 + 2px^2 - px + 4$, where p is a constant.

If we divide the polynomial by $x - h$, then we will obtain an expression for the remainder in terms of the unknown constants.

If we already know the value of the remainder, we can solve for the unknown constants.

EXAMPLES

1. Given that $x - 3$ is a factor of $x^3 - x^2 + px + 24$, find the value of p.

$x - 3$ is a factor $\Leftrightarrow$ $x = 3$ is a root.

$$\begin{array}{r|rrrr} 3 & 1 & -1 & p & 24 \\ & & 3 & 6 & 18+3p \\ \hline & 1 & 2 & 6+p & \boxed{42+3p} \end{array}$$

Since $x - 3$ is a factor, the remainder is zero:

$$\begin{aligned} 42 + 3p &= 0 \\ 3p &= -42 \\ p &= -14 \end{aligned}$$

2. When $f(x) = px^3 + qx^2 - 17x + 4q$ is divided by $x - 2$, the remainder is 6, and $x - 1$ is a factor of $f(x)$.

Find the values of p and q.

When $f(x)$ is divided by $x - 2$, the remainder is $f(2)$:

$$\begin{array}{c|cccc} 2 & p & q & -17 & 4q \\ & & 2p & 4p+2q & 8p+4q-34 \\ \hline & p & 2p+q & 4p+2q-17 & \boxed{8p+8q-34} \end{array}$$

Since $f(2) = 6$, we have:

$$\begin{aligned} 8p + 8q - 34 &= 6 \\ 8p + 8q &= 40 \\ p + q &= 5 \qquad (1) \end{aligned}$$

Since $x - 1$ is a factor, $f(1) = 0$:

$$\begin{aligned} f(1) &= p(1)^3 + q(1)^2 - 17(1) + 4q \\ &= p + q - 17 + 4q \\ &= p + 5q - 17 \end{aligned}$$

i.e. $p + 5q - 17 = 0 \qquad (2)$

Note
There is no need to use synthetic division here

Solving (1) and (2) simultaneously, we obtain:

$$\begin{aligned} p + 5q &= 17 \qquad (2) \\ -p - q &= -5 \qquad (1)\times -1 \\ \hline 4q &= 12 \\ q &= 3 \end{aligned}$$

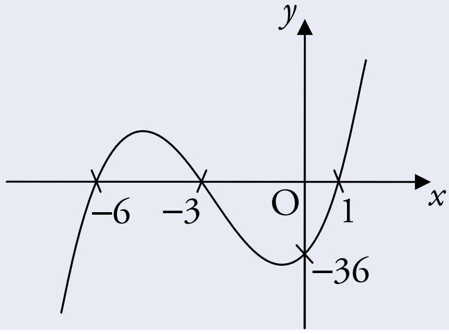
Giving $p = 5 - q = 5 - 3 = 2$ from (1). Hence $p = 2$ and $q = 3$.

11 Determining the Equation of a Graph

Given the roots, and at least one other point lying on the graph, we can establish the graph's equation.

EXAMPLE

1. Find the equation of the cubic shown in the diagram below.



Step 1

Write out the roots, then rearrange to get the factors.

$$x = -6 \qquad x = -3 \qquad x = 1$$
$$x + 6 = 0 \qquad x + 3 = 0 \qquad x - 1 = 0$$

Step 2

The equation is then these factors multiplied together with a constant, k.

$$y = k(x+6)(x+3)(x-1)$$

Step 3

Substitute the coordinates of a known point into this equation to find the value of k.

Using $(0, -36)$:

$$\begin{aligned} k(0+6)(0+3)(0-1) &= -36 \\ k(3)(-1)(6) &= -36 \\ -18k &= -36 \\ k &= 2 \end{aligned}$$

Step 4

Replace k with this value in the equation.

$$\begin{aligned} y &= 2(x+6)(x+3)(x-1) \\ &= 2(x+3)\left(x^2+5x-6\right) \\ &= 2\left(x^3+5x^2-6x+3x^2+15x-18\right) \\ &= 2x^3+16x^2+18x-36 \end{aligned}$$

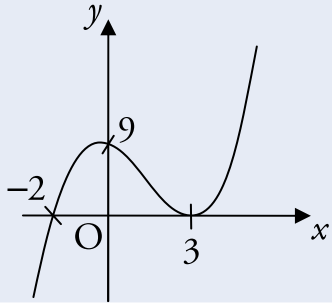
Repeated Roots

If a repeated root exists, then a stationary point lies on the x-axis.

Recall that a repeated root exists when two roots, and hence two factors, are equal.

EXAMPLE

2. Find the equation of the cubic shown in the diagram below.



$$x=-2 \qquad x=3 \qquad x=3$$

$$x+2=0 \qquad x-3=0 \qquad x-3=0$$

So $y=k(x+2)(x-3)^2$.

Use $(0,9)$ to find k:

$$9=k(0+2)(0-3)^2$$

$$9=k\times2\times9$$

$$k=\tfrac{1}{2}$$

So
$$\begin{aligned} y&=\tfrac{1}{2}(x+2)(x-3)^2 \\ &=\tfrac{1}{2}(x+2)\left(x^2-6x+9\right) \\ &=\tfrac{1}{2}\left(x^3-6x^2+9x+2x^2-12x+18\right) \\ &=\tfrac{1}{2}x^3-2x^2-\tfrac{3}{2}x+9 \end{aligned}$$

12 Iteration

When a root of a polynomial is not rational, an estimate of the root can be determined using an iterative process.

EXAMPLE

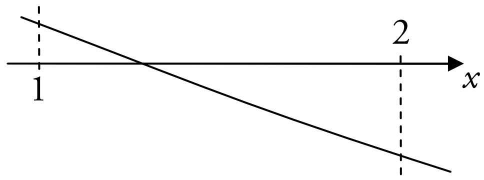
If $f(x) = x^3 - 4x^2 - 2x + 7$, show that there is a real root between $x = 1$ and $x = 2$. Find this root correct to two decimal places.

Evaluate $f(x)$ at $x = 1$ and $x = 2$:

$$f(1) = 1^3 - 4(1)^2 - 2(1) + 7 = 2$$

$$f(2) = 2^3 - 4(2)^2 - 2(2) + 7 = -5$$

Since $f(1) > 0$ and $f(2) < 0$, $f(x)$ has a root between $x = 1$ and $x = 2$.



Try halfway between $x = 1$ and $x = 2$:

$$f(1{\cdot}5) = 1{\cdot}5^3 - 4(1{\cdot}5)^2 - 2(1{\cdot}5) + 7 = -1{\cdot}625 < 0$$

$f(1) > 0$ and $f(1{\cdot}5) < 0$ so the root is between $x = 1$ and $x = 1{\cdot}5$.

Try roughly halfway between $x = 1$ and $x = 1{\cdot}5$:

$$f(1{\cdot}3) = 1{\cdot}3^3 - 4(1{\cdot}3)^2 - 2(1{\cdot}3) + 7 = -0{\cdot}163 < 0$$

$$f(1{\cdot}2) = 1{\cdot}2^3 - 4(1{\cdot}2)^2 - 2(1{\cdot}2) + 7 = 0{\cdot}568 > 0$$

$f(1{\cdot}2) > 0$ and $f(1{\cdot}3) < 0$ so the root is between $x = 1{\cdot}2$ and $x = 1{\cdot}3$.

Try halfway between $x = 1{\cdot}2$ and $x = 1{\cdot}3$:

$$f(1{\cdot}25) = 1{\cdot}25^3 - 4(1{\cdot}25)^2 - 2(1{\cdot}25) + 7 = 0{\cdot}203125 > 0$$

$f(1{\cdot}25) > 0$ and $f(1{\cdot}3) < 0$ so the root is between $x = 1{\cdot}25$ and $x = 1{\cdot}3$.

Try roughly halfway between $x = 1{\cdot}25$ and $x = 1{\cdot}3$:

$$f(1{\cdot}27) = 1{\cdot}27^3 - 4(1{\cdot}27)^2 - 2(1{\cdot}27) + 7 = 0{\cdot}056783 > 0$$

$$f(1{\cdot}28) = 1{\cdot}28^3 - 4(1{\cdot}28)^2 - 2(1{\cdot}28) + 7 = -0{\cdot}016448 < 0$$

$f(1{\cdot}27) > 0$ and $f(1{\cdot}28) < 0$ so the root is between $x = 1{\cdot}27$ and $x = 1{\cdot}28$.

Try halfway between $x = 1{\cdot}27$ and $x = 1{\cdot}28$:

$$f(1.275) = 1{\cdot}275^3 - 4(1{\cdot}275)^2 - 2(1{\cdot}275) + 7 = 0.020171875$$

$f(1{\cdot}275) > 0$ and $f(1{\cdot}28) < 0$ so the root is between $x = 1{\cdot}275$ and $x = 1{\cdot}28$.

Therefore the root is $x = 1{\cdot}28$ to 2 d.p.