OUTCOME 1
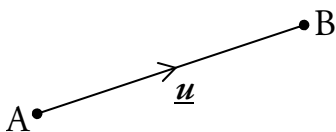# Vectors

1   Vectors and Scalars

A **scalar** is a quantity with magnitude (size) only – for example, an amount of money or a length of time.

Sometimes size alone is not enough to describe a quantity – for example, directions to the nearest shop. For this we need to know a magnitude (i.e. how far), *and* a direction. Quantities with magnitude and direction are called **vectors**.

A vector is named either by using the letters at the end of a directed line segment (e.g. $\overrightarrow{AB}$) or by using a bold letter (e.g. *u*). You will see bold letters used in printed text, however we show this as u̲ when we write it (i.e. we underline the letter).

Throughout these notes, we will show vectors in bold as well us underlining them (e.g. *u̲*).

2   Components

A vector may be represented by its **components**. These define how to get from one end of the vector to the other. For example:

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a vector in two dimensions $\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$ is a vector in three dimensions

The components are written in a column, to avoid confusing vectors with coordinates.

Zero Vectors

Any vector with all its components zero is called a zero vector and can be written as $\underline{0}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$.

3   Magnitude

The magnitude (length) of a vector is written as $|\underline{u}|$ or $|\overrightarrow{AB}|$. It can be calculated as follows:

$$\text{If } \overrightarrow{PQ} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{then} \quad |\overrightarrow{PQ}| = \sqrt{a^2 + b^2}$$

$$\text{If } \overrightarrow{PQ} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{then} \quad |\overrightarrow{PQ}| = \sqrt{a^2 + b^2 + c^2}$$

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, find $|\underline{u}|$.

$$|\underline{u}| = \sqrt{(5)^2 + (-12)^2}$$
$$= \sqrt{169}$$
$$= 13 \text{ units}$$

2. Find the length of $\underline{a} = \begin{pmatrix} -\sqrt{5} \\ 6 \\ 3 \end{pmatrix}$.

$$|\underline{a}| = \sqrt{\left(-\sqrt{5}\right)^2 + 6^2 + 3^2}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2} \text{ units}$$

Unit Vectors

Any vector with a magnitude of one is called a unit vector. For example:

$$\text{If } \underline{u} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \text{ then } |\underline{u}| = \sqrt{\left(\tfrac{1}{2}\right)^2 + 0^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}$$
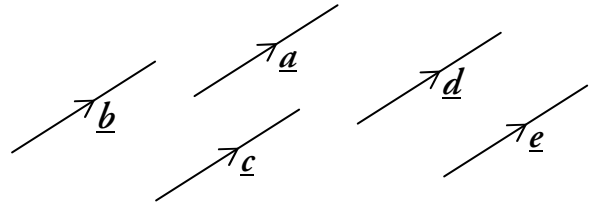$$= \sqrt{\tfrac{4}{4}}$$
$$= 1 \text{ unit}$$

So $\underline{u}$ is a unit vector.

4   Equal Vectors

Vectors with the same magnitude and direction are equal.

For example, all the vectors shown to the right are equal.

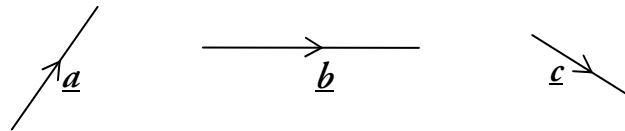If vectors are equal to each other, then all of their components are equal, i.e.

$$\text{If } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \quad \text{then} \quad a = d, \ b = e \text{ and } c = f$$

The converse is also true.

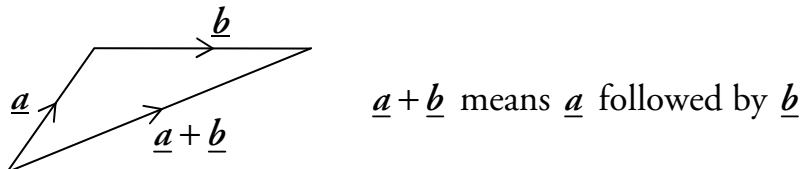# 5   Addition and Subtraction of Vectors
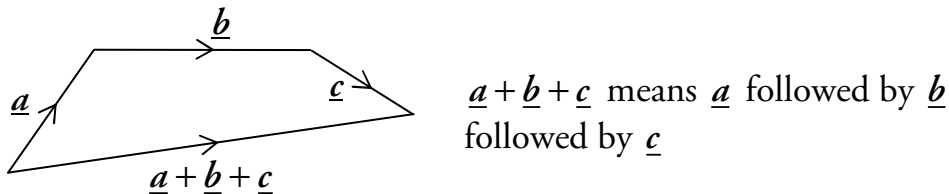
Consider the following vectors:

Addition

We can construct $\underline{a} + \underline{b}$ as follows:

$\underline{a} + \underline{b}$ means $\underline{a}$ followed by $\underline{b}$

Similarly, we can construct $\underline{a} + \underline{b} + \underline{c}$ as follows:

$\underline{a} + \underline{b} + \underline{c}$ means $\underline{a}$ followed by $\underline{b}$ followed by $\underline{c}$

To add vectors, we position them nose-to-tail. Then the sum of the vectors is the vector between the first tail and the last nose.

Subtraction

Now consider $\underline{a} - \underline{b}$. This can be written as $\underline{a} + (-\underline{b})$, so if we first find $-\underline{b}$, we can use vector addition to obtain $\underline{a} - \underline{b}$.

$-\underline{b}$ is just $\underline{b}$ but in the opposite direction.

$-\underline{b}$ and $\underline{b}$ have the same magnitude, i.e. $\left|\underline{b}\right| = \left|-\underline{b}\right|$.

Therefore we can construct $\underline{a} - \underline{b}$ as follows:

$\underline{a} - \underline{b}$ means $\underline{a}$ followed by $-\underline{b}$
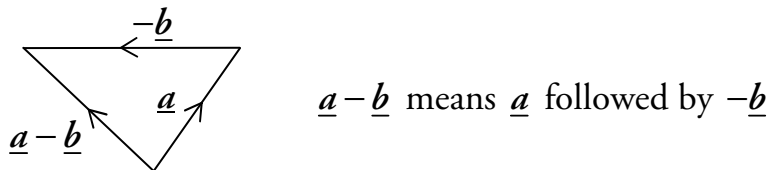
Using Components

If we have the components of vectors, then things become much simpler.

The following rules can be used for addition and subtraction.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix} \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$$

add the components   subtract the components

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, calculate $\underline{u} + \underline{v}$ and $\underline{u} - \underline{v}$.

$$\underline{u} + \underline{v} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \qquad \underline{u} - \underline{v} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

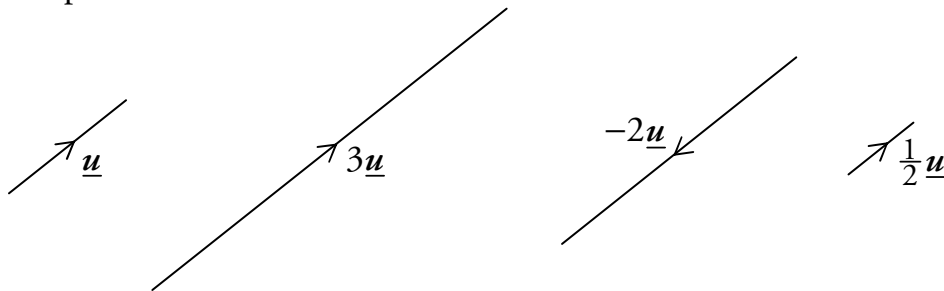$$= \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix} \qquad\qquad = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

2. Given $\underline{p} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}$, calculate $\underline{p} - \underline{q}$ and $\underline{q} + \underline{p}$.

$$\underline{p} - \underline{q} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix} \qquad \underline{q} + \underline{p} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -\frac{3}{2} \\ \frac{21}{5} \end{pmatrix} \qquad\qquad\qquad = \begin{pmatrix} 3 \\ \frac{9}{2} \\ \frac{9}{5} \end{pmatrix}$$

6 Multiplication by a Scalar

A vector $\underline{u}$ which is multiplied by a scalar k will give the result $k\underline{u}$. This vector will be k times as long, i.e. the magnitude will be $k|\underline{u}|$.

For example:

$\underline{u}$ $3\underline{u}$ $-2\underline{u}$ $\frac{1}{2}\underline{u}$

If $\underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $k\underline{u} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$

Each component is multiplied by the scalar.

EXAMPLES

1. Given $\underline{v} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$, find $3\underline{v}$.

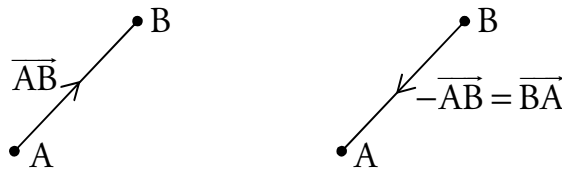$$3\underline{v} = 3\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \\ -9 \end{pmatrix}$$

2. Given $\underline{r} = \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}$, find $-4\underline{r}$.

$$-4\underline{r} = -4\begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -12 \\ -4 \end{pmatrix}$$

Negative Vectors

The negative of a vector is the vector multiplied by -1.

If we write a vector as a directed line segment $\overrightarrow{AB}$, then $-\overrightarrow{AB} = \overrightarrow{BA}$:
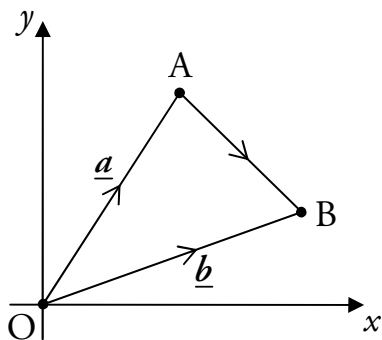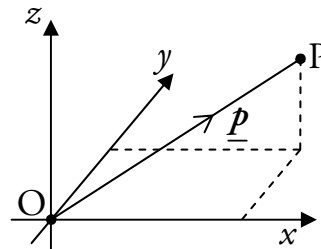

7   Position Vectors

$\overrightarrow{OA}$ is called a position vector of point A relative to the origin O, and is written as $\underline{a}$.

$\overrightarrow{OB}$ is called the position vector of point B, written $\underline{b}$.

Given $P(x, y, z)$, the position vector

$\overrightarrow{OP}$ or $\underline{p}$ has components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

To move from point A to point B we can move back along the vector $\underline{a}$ to the origin, and along vector $\underline{b}$ to point B.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\overrightarrow{OA} + \overrightarrow{OB}$$
$$= -\underline{a} + \underline{b}$$
$$= \underline{b} - \underline{a}$$

For the vector joining any two points P and Q, $\overrightarrow{PQ} = \underline{q} - \underline{p}$.

EXAMPLE

R is the point $(2, -2, 3)$ and S is the point $(4, 6, -1)$. Find $\overrightarrow{RS}$.

From the coordinates, $\underline{r} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and $\underline{s} = \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix}$.
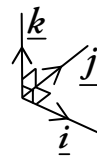
Note
There is no need to write this down

$$\overrightarrow{RS} = \underline{s} - \underline{r}$$

$$= \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}$$

8   Basis Vectors

A vector may also be defined in terms of the basis vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$.
These are three mutually perpendicular unit vectors (i.e. they are perpendicular to each other).

These basis vectors can be written in component form as

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector can be written in **basis form** using $\underline{i}$, $\underline{j}$ and $\underline{k}$. For example:

$$\underline{v} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\underline{i} - 3\underline{j} + 6\underline{k}$$

There is no need for the working above if the following is used:

$$a\underline{i} + b\underline{j} + c\underline{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

9   Collinearity

In Straight Lines (Unit 1 Outcome 1), we learned that points are collinear if they lie on the same straight line.

not collinear   not collinear   collinear
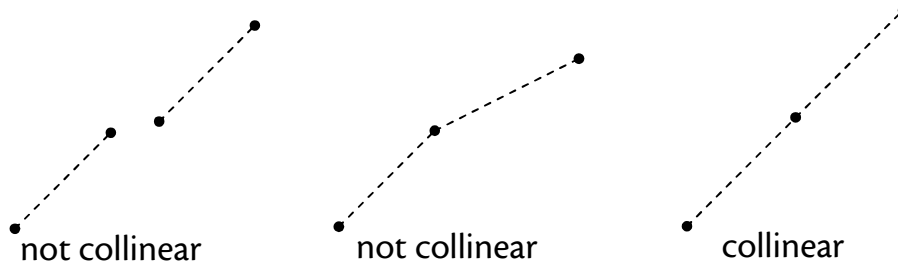
> The points A, B and C in 3D space are collinear if $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, with B a common point.

Note that we **cannot** find gradients in three dimensions – instead we use the fact that vectors are parallel if they are scalar multiples of the same vector.

For example:

$$\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \qquad \underline{v} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 3\underline{u}$$

So $\underline{u}$ and $\underline{v}$ are parallel.

$$\underline{p} = \begin{pmatrix} 15 \\ 9 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}, \qquad \underline{q} = \begin{pmatrix} 20 \\ 12 \\ -8 \end{pmatrix} = 4 \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$$

So $\underline{p}$ and $\underline{q}$ are parallel.

EXAMPLE

A is the point $(1, -2, 5)$, B$(8, -5, 9)$ and C$(22, -11, 17)$.

Show that A, B and C are collinear.

$$\overrightarrow{AB} = \underline{b} - \underline{a} \qquad\qquad\qquad \overrightarrow{BC} = \underline{c} - \underline{b}$$

$$= \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \qquad\qquad = \begin{pmatrix} 22 \\ -11 \\ 17 \end{pmatrix} - \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix} \qquad\qquad\qquad = \begin{pmatrix} 14 \\ -6 \\ 8 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}$$

$\overrightarrow{BC} = 2\overrightarrow{AB}$, so $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel – and since B is a common point, A, B and C are collinear.

10 Dividing Lines in a Ratio

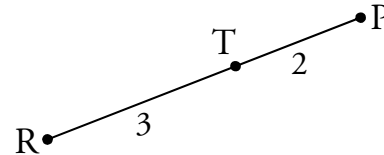There is a simple process for finding the coordinates of a point which divides a line segment in a given ratio.

EXAMPLE

1. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

The point T divides PR in the ratio $2:3$. Find the coordinates of T.

Step 1
Make a sketch of the line, showing the
ratio in which the point divides the line.

Step 2
Using the sketch, equate the ratio of the
two lines with the given ratio.

$$\frac{PT}{TR} = \frac{2}{3}$$

Step 3
Cross multiply, then change directed
line segments to position vectors.

$$3\overrightarrow{PT} = 2\overrightarrow{TR}$$

$$3\left(\underline{t} - \underline{p}\right) = 2\left(\underline{r} - \underline{t}\right)$$

Step 4
Rearrange to give the position vector of
the unknown point.

$$3\underline{t} - 3\underline{p} = 2\underline{r} - 2\underline{t}$$

$$3\underline{t} + 2\underline{t} = 2\underline{r} + 3\underline{p}$$

$$5\underline{t} = 2\begin{pmatrix} 8 \\ -1 \\ 19 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix} 16 \\ -2 \\ 38 \end{pmatrix} + \begin{pmatrix} -6 \\ 12 \\ -3 \end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix} 10 \\ 10 \\ 35 \end{pmatrix}$$

$$\underline{t} = \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}$$

Step 5
From the position vector, state the
coordinates of the unknown point.

So T is the point $(2, 2, 7)$.

Using the Section Formula

The previous method can be condensed into a formula as shown below.

If the point P divides the line AB in the ratio $m:n$, then:

$$\underline{p} = \frac{n\underline{a} + m\underline{b}}{n + m}$$

where is $\underline{a}$, $\underline{b}$ and $\underline{p}$ are the position vectors of A, B and P.

This is referred to as the **section formula**.

It is not necessary to know this, since the approach explained above will always work.

EXAMPLE

2. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

    The point T divides PR in the ratio $2:3$. Find the coordinates of T.

    The ratio is $2:3$, so let $m = 2$ and $n = 3$, then:

$$\underline{t} = \frac{n\underline{p} + m\underline{r}}{n + m}$$

$$= \frac{3\underline{p} + 2\underline{r}}{5}$$

$$= \begin{pmatrix} \frac{1}{5}[3(-2) + 2(8)] \\ \frac{1}{5}[3(4) + 2(-1)] \\ \frac{1}{5}[3(-1) + 2(19)] \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}$$

Note
If you are confident with arithmetic, this step can be done mentally

So T is the point $(2, 2, 7)$.

11 The Scalar Product

So far we have added and subtracted vectors and multiplied a vector by a scalar. Now we will consider the scalar product, which is a form of vector multiplication.
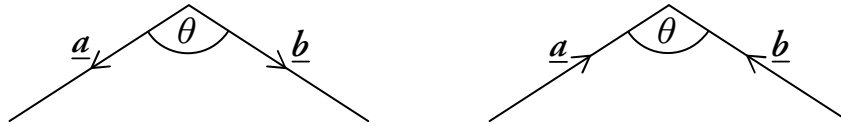
> The scalar product is defined as $\underline{a}.\underline{b}$ (sometimes it is called the dot product) and can be calculated as follows:
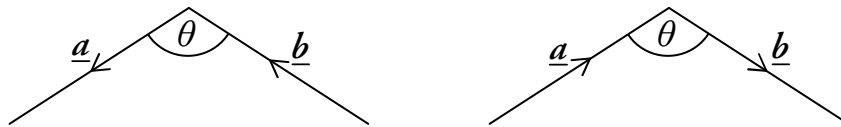> $$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$$
> where θ is the angle between the two vectors $\underline{a}$ and $\underline{b}$.

This is given on the formulae list.

The definition above assumes that the vectors $\underline{a}$ and $\underline{b}$ are positioned so that they both point away from the angle, or both point into the angle.
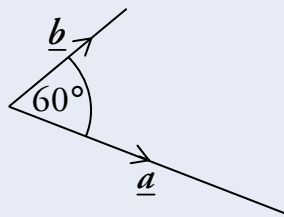

However, if one vector is pointing away from the angle, while the other points into the angle

we find that $\underline{a}.\underline{b} = -|\underline{a}||\underline{b}|\cos\theta$.

EXAMPLE

1. Two vectors, $\underline{a}$ and $\underline{b}$ have magnitudes 7 and 3 units respectively and are at an angle of $60°$ to each other as shown below.

   What is the value of $\underline{a}.\underline{b}$?

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$$
$$= 7 \times 3 \times \cos 60°$$
$$= 21 \times \tfrac{1}{2}$$
$$= \tfrac{21}{2}$$

The Component Form of the Scalar Product

The scalar product can also be calculated as follows:

$$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 \quad \text{where } \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$
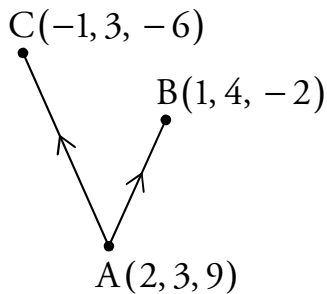
This is given on the formulae list.

EXAMPLES

2. Find $\underline{p}.\underline{q}$, given that $\underline{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

$$\begin{aligned}
\underline{p}.\underline{q} &= p_1q_1 + p_2q_2 + p_3q_3 \\
&= (1\times2)+(2\times2)+((-3)\times3) \\
&= 2+4-9 \\
&= -3
\end{aligned}$$

3. If A is the point $(2,3,9)$, $B(1,4,-2)$ and $C(-1,3,-6)$, calculate $\overrightarrow{AB}.\overrightarrow{AC}$.

$C(-1,3,-6)$

$B(1,4,-2)$

$A(2,3,9)$

We need to use the position vectors of the points:

$$\overrightarrow{AB} = \underline{b} - \underline{a} \qquad\qquad \overrightarrow{AC} = \underline{c} - \underline{a}$$

$$= \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} \qquad\qquad = \begin{pmatrix} -1 \\ 3 \\ -6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ -11 \end{pmatrix} \qquad\qquad = \begin{pmatrix} -3 \\ 0 \\ -15 \end{pmatrix}$$

$$\begin{aligned}
\overrightarrow{AB}.\overrightarrow{AC} &= ((-1)\times(-3))+(1\times0)+((-11)\times(-15)) \\
&= 3+0+165 \\
&= 168
\end{aligned}$$

12 The Angle Between Vectors

The scalar product can be rearranged to give the following equations, both of which can be used to calculate θ, the angle between the two vectors.

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} \qquad \text{or} \qquad \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\underline{a}||\underline{b}|}$$

Look back to the formulae for finding the scalar product, given on the previous two pages. Notice that the first equation is simply a rearranged form of the one which can be used to find the scalar product. Also notice that the second simply substitutes $\underline{a}.\underline{b}$ for one of its equivalences.

These formulae are **not** given in the exam but can both be easily derived from the formulae on the previous pages (which are given).

EXAMPLES

1. Calculate the angle θ between vectors $\underline{p} = 3\underline{i} + 4\underline{j} - 2\underline{k}$ and
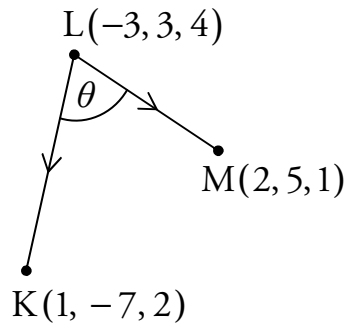 $\underline{q} = 4\underline{i} + \underline{j} + 3\underline{k}$.

$$\underline{p} = 3\underline{i} + 4\underline{j} - 2\underline{k} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \text{ and } \underline{q} = 4\underline{i} + \underline{j} + 3\underline{k} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\cos\theta = \frac{p_1q_1 + p_2q_2 + p_3q_3}{|\underline{p}||\underline{q}|}$$

$$= \frac{(3\times4) + (4\times1) + ((-2)\times3)}{\sqrt{3^2 + 4^2 + (-2)^2}\sqrt{4^2 + 1^2 + 3^2}}$$

$$= \frac{10}{\sqrt{29}\sqrt{26}}$$

$$\theta = \cos^{-1}\left(\frac{10}{\sqrt{29}\sqrt{26}}\right)$$

$$= 68\cdot6° \text{ (to 1 d.p.)} \qquad \text{(or } 1\cdot198 \text{ radians (to 3 d.p.))}$$

2. K is the point $(1, -7, 2)$, $L(-3, 3, 4)$ and $M(2, 5, 1)$. Find $K\hat{L}M$.

Start with a sketch:

$\overrightarrow{LK} = \underline{k} - \underline{l}$

$\overrightarrow{LM} = \underline{m} - \underline{l}$

$= \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -10 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$

$\cos\theta = \dfrac{\overrightarrow{LK}.\overrightarrow{LM}}{\left|\overrightarrow{LK}\right|\left|\overrightarrow{LM}\right|}$

$= \dfrac{(4 \times 5) + (-10 \times 2) + (-2 \times (-3))}{\sqrt{4^2 + (-10)^2 + (-2)^2}\sqrt{5^2 + 2^2 + (-3)^2}}$

$= \dfrac{6}{\sqrt{120}\sqrt{38}}$

$\theta = \cos^{-1}\left(\dfrac{6}{\sqrt{120}\sqrt{38}}\right)$

$= 84 \cdot 9°$ (to 1 d.p.)   (or $1 \cdot 482$ radians (to 3 d.p.))

13 Perpendicular Vectors

If $\underline{a}$ and $\underline{b}$ are perpendicular then $\underline{a}.\underline{b} = 0$.

This is because $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$$= |\underline{a}||\underline{b}|\cos 90° \qquad (\theta = 90° \text{ since perpendicular})$$

$$= 0 \qquad\qquad\qquad (\text{since } \cos 90° = 0)$$

Conversely, if $\underline{a}.\underline{b} = 0$ then $\underline{a}$ and $\underline{b}$ are perpendicular.

EXAMPLES

1. Two vectors are defined as $\underline{a} = 4\underline{i} + 2\underline{j} - 5\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 2\underline{k}$.

    Show that $\underline{a}$ and $\underline{b}$ are perpendicular.

    $\underline{a}.\underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

    $\quad = (4 \times 2) + (2 \times 1) + ((-5) \times 2)$

    $\quad = 8 + 2 - 10$

    $\quad = 0$

    Since $\underline{a}.\underline{b} = 0$, $\underline{a}$ and $\underline{b}$ are perpendicular.

2. $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ a \\ 7 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \\ a \end{pmatrix}$ where a is a constant.

    Given that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are perpendicular, find the value of a.

    $$\overrightarrow{PQ}.\overrightarrow{RS} = 0 \qquad\qquad \text{Since } \overrightarrow{PQ} \text{ and } \overrightarrow{RS} \text{ are perpendicular}$$

    $$4 \times 2 + (-3a) + 7a = 0$$
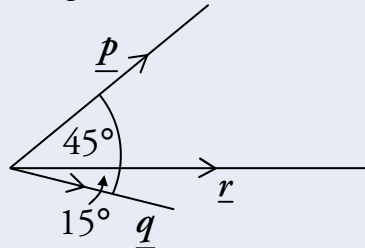
    $$8 - 3a + 7a = 0$$

    $$8 + 4a = 0$$

    $$a = -2$$

14 Properties of the Scalar Product

Some properties of the scalar product are as follows:

$$\underline{a}.\underline{b} = \underline{b}.\underline{a}$$

$$\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c} \qquad \text{(Expanding brackets)}$$
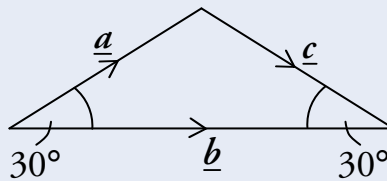
$$\underline{a}.\underline{a} = |\underline{a}|^2$$

EXAMPLES

1. Calculate $\underline{p}.(\underline{q}+\underline{r})$ when $|\underline{p}|=3$, $|\underline{r}|=4$ and $|\underline{q}|=2$ and the vectors are arranged as shown in the diagram below.

$$\underline{p}.(\underline{q}+\underline{r}) = \underline{p}.\underline{q} + \underline{p}.\underline{r}$$

$$= |\underline{p}||\underline{q}|\cos\theta_1 + |\underline{p}||\underline{r}|\cos\theta_2$$

$$= 3\times 2\times\cos 60° + 3\times 4\times\cos 45°$$

$$= 6\times\tfrac{1}{2} + 12\times\tfrac{1}{\sqrt{2}}$$

$$= 3 + 6\sqrt{2}$$

2. In the diagram below $|\underline{a}| = |\underline{c}| = 2$ and $|\underline{b}| = 2\sqrt{3}$.

Calculate $\underline{a}.(\underline{a}+\underline{b}+\underline{c})$.

$$\underline{a}.(\underline{a}+\underline{b}+\underline{c})$$

$$= \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{a}.\underline{c}$$

$$= |\underline{a}|^2 + |\underline{a}||\underline{b}|\cos\theta_1 - |\underline{a}||\underline{c}|\cos\theta_2$$

$$= 2^2 + 2\times 2\sqrt{3}\times\cos 30° - 2\times 2\times\cos 120°$$

$$= 4 + 4\sqrt{3}\times\tfrac{\sqrt{3}}{2} + 4\times\tfrac{1}{2}$$

$$= 4 + 6 + 2$$

$$= 12$$

Remember
$\underline{a}.\underline{c} = -|\underline{a}||\underline{c}|\cos\theta_2$
since $\underline{a}$ points into θ_2
and $\underline{c}$ points away